Stretched random walks and the behaviour of their summands.

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Abstract

This paper explores the joint behaviour of the summands of a random walk when their mean value goes to infinity as its length increases. It is proved that all the summands must share the same value, which extends previous results in the context of large exceedances of finite sums of i.i.d. random variables. Some consequences are drawn pertaining to the local behaviour of a random walk conditioned on a large deviation constraint on its end value. It is shown that the sample paths exhibit local oblic segments with increasing size and slope as the length of the random walk increases.

Key words: Random Walk, Extreme deviation, Large deviation, Erdös-Rényi law of large numbers, democratic localization.

1 Introduction

1.1 Context and scope

This paper considers the following question: Let $X, X_1, .., X_n$ denote real valued independent random variables (r.v's) distributed as X and let $S_1^n := X_1 + .. + X_n$. We assume that X is unbounded upwards. Let a_n be some positive sequence satisfying

$$\lim_{n\to\infty} a_n = +\infty. \tag{1}$$

Assuming that

$$C := (S_1^n/n > a_n) \tag{2}$$

holds, what can be inferred on the r.v's X_i's as n goes to infinity?

Let ε_n denote a positive sequence and let

$$I := \cap_{i=1}^{n} \left(X_i \in (a_n - \epsilon_n, a_n + \epsilon_n) \right). \tag{3}$$

We consider cases when

$$\lim_{n \to \infty} P\left(I | C \right) = 1. \tag{4}$$

The relation between the various parameters in this problem is of interest and opens a variety of questions. For which distributions P_X pertaining to X is such a result valid? Which is the acceptable growth of the sequence a_n and the possible behaviours of the sequence ε_n such that

$$\epsilon_n = o\left(a_n\right) \tag{5}$$

and is it possible to achieve

$$\lim_{n \to \infty} \epsilon_n = 0 \tag{6}$$

under a large class of choices for P_X?

In the case when the r.v. X has light tails conditional limit theorems exploring the behavior of the summands of a random walk given its sum have been developped extensively in the range of a large deviation conditioning event, namely similar as defined by C with fixed a_n, hence lower-bounding S_n/n independently on n; the papers [9], or [12] together with their extension in [11] explore the asymptotic properties of a relatively small number of summands; the main result in these papers, named as Gibbs conditional principle, lies in the fact that under such C, the X_i's are asymptotically i.i.d. with distribution Π^a defined through $d\Pi^a(x) := \left(E\left(\exp tX\right)\right)^{-1} \exp(tx) dP_X(x)$ where t satisfies $E\left(X \exp tX\right)\left(E\left(\exp tX\right)\right)^{-1} = a$; in this range (6) does not hold. The joint distribution of $X_1, .., X_{k_n}$ given C (with fixed a_n) for large k_n (close to n) is considered in [6] .

Extended large deviations results for $a_n \to \infty$ have been considered in [5], [8], in relation with versions of the Erdös-Rényi law of large numbers for the small increments of a random walk, and [16].

The case when X is heavy tailed is considered in [1] where the authors consider the support of the distribution of the whole sample $X_1, .., X_n$ when C holds for fixed a_n .

A closely related problem has been handled by statisticians in various contexts, exploring the number of sample observations which push a given statistics far away from its expectation, for *fixed* n. Although similar in phrasing as the so-called "breakdown point" paradigm of robust

analysis , the frame of this question is quite different from the robustness point of view, since all the observations are supposed to be sampled under the distribution P_X, hence without any reference to outliers or misspecification. The question may therefore be stated as: how many sample points should be large making a given statistics large? This combines both the asymptotic behavior of the statistics (as a function defined on $\mathbb{R}^n$) and the tail properties of P_X. In the case when the statistics is S_1^n/n and X has subexponential upper tail, it is well known that, denoting

$$C_a := (S_1^n/n > a)$$

only one large value of the X_i's generates C_a for $a \to \infty$; clearly S_n/n is not a loyal statistics under this sampling. This result turns back to Darling (1952). For light tails, under C_a, all sampled values should exceed a (indeed they should be closer and closer to a as $a \to \infty$), so that S_n/n is faithfull in allegeance with respect to the sample. In this case, denoting

$$I_a := \cap_{i=1}^n (X_i > a)$$

it holds

$$\lim_{a\to\infty} P\left(I_a | C_a\right) = 1 \tag{7}$$

Intermediate cases exist, leading to partial loyalty for a given statistics under a given sampling scheme. See [7], [3], and [2] where more general statistics than S_n/n are considered. and $a \to \infty$. According to the tail behavior of the distribution of X the situation may take quite different features.

Related questions have also been considered in the realm of statistical physics. In [14] the property (7) is stated in an improved form, namely stating that when the X_i's are i.i.d. with Weibull density with shape index larger than 2 then the conditional density of $(X_1, .., X_n)$ given $(S_1^n/n = a)$ concentrates at $(a, ..a)$ as $a \to \infty$, which in the authors' words means that the X_i's are *democratically localized*. Applications of this concept in fragmentation processes, in some form of anomalous relaxation of glasses and in the study of turbulence flows are discussed.

We now come to a consequence of the present results considering the local behaviour of a random walk conditioned on its end value. Let $S_i^j := X_i + .. + X_j$ with $1 \leq i \leq j \leq n$ and $k = k_n$ denote an integer valued sequence such that

$$k_n \leq n$$

and

$$\lim_{n\to\infty} k_n = \infty.$$

Let further

$$\Delta_{j,n} := S_{j+1}^{j+k}/k$$

denote the local slope of the random walk on the interval $[j+1, j+k]$ where $1 \leq j \leq n-k$. The limit behaviour of $\max_{1\leq j\leq n-k} \Delta_{j,n}$ has been considered extensively in various cases, according to the order of magnitude of k. The case $k = C \log n$ for positive constant C defines the so-called Erdös-Rényi law of large numbers; see [13]. In the present case we consider random walks conditioned upon their end value, namely assuming that

$$S_1^n > na$$

for fixed $a > EX$. We will prove that as $n \to \infty$ the path defined by this random walk exhibits anomalous local behavior that can be captured through the *extended democratic localization principle* stated in our results. Indeed there exist segments of length k_n on which the slope $\Delta_{j,n}$ tends to infinity with a rate which can be made precise. Simulations are proposed in order to enlight this phenomenon. Obviously, when a is not fixed but goes to infinity with n then the *extended democratic localization principle* applies to the whole sample path of the random walk, and its trajectory is nearly a stright line from the origin up to its extremity. When conditioning in the range of the large deviation only, this property holds locally.

This paper is organized as follows. Section 2 states the notation and hypotheses. Section 3 states the results in two cases; the first one pertains to the case when X has a log-concave density and the second case is a generalizetion of the former. Examples are, provided. Section 4 presents a short account on the local behaviour of random paths from conditioned random walk, with some simulation. The proofs of the results are rather long and technical; they have been postponed to the Appendix.

2 Notation and hypotheses

The n real valued random variables $X_1, ..., X_n$. are independent copies of a r.v. X with density p whose support is $\mathbb{R}^+$. As seen by the very nature of the problem handled in this paper, this assumption puts no restriction to the results. We write

$$p(x) := \exp -h(x)$$

for positive functions h which are defined and denoted according to the context. For $\mathbf{x} \in \mathbb{R}^n$ define

$$I_h(\mathbf{x}) := \sum_{1\leq i\leq n} h(x_i),$$

and for A a Borel set in $\mathbb{R}^n$ denote

$$I_h(A) = \inf_{(\mathbf{x})\in A} I_h\left(\mathbf{x}\right).$$

Two cases will be considered: in the first one h is assumed to be a convex function, and in the second case h will be the sum of a convex function and a "smaller" function h in such a way that we will also handle non log-concave densities.(although not too far from them). Hence we do not consider heavy tailed r.v. X.

For positive r define

$$S(r) = \left\{ \mathbf{x} := (x_1, .., x_n) : \sum_{1\leq i\leq n} h(x_i) \leq r \right\}.$$

3 Very Large Deviation for Exponential Density Functions associated to Convex Functions

Lemma 1 *Let g be a positive convex differentiable function defined on $\mathbb{R}_+$. Assume that g is strictly increasing on some interval $[X, \infty)$. Let (1) hold. Then*

$$I_g(I^c \cap C) = \min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right),$$

where

$$F_{g_1}(a_n, \epsilon_n) = g(a_n + \epsilon_n) + (n-1)g\left(a_n - \frac{1}{n-1}\epsilon_n\right),$$

and

$$F_{g_2}(a_n, \epsilon_n) = g(a_n - \epsilon_n) + (n-1)g\left(a_n + \frac{1}{n-1}\epsilon_n\right).$$

Theorem 2 *Let $X_1, ..., X_n$ be i.i.d. copies of a r.v. X with density $p(x) = c\exp\left(-g(x)\right)$, where $g(x)$ is a positive convex function on $\mathbb{R}^+$. Assume that g is increasing on some interval $[X, \infty)$ and satisfies*

$$\lim_{x\to\infty} g(x)/x = \infty.$$

Let a_n satisfy

$$\lim\inf_{n\to\infty} \frac{\log a_n}{\log n} n > 0$$

and that for some positive sequence ϵ_n

$$\lim_{n\to\infty} \frac{n \log g\left(a_n + \epsilon_n\right)}{H(a_n, \epsilon_n)} = 0, \tag{8}$$

$$\lim_{n\to\infty} \frac{nG(a_n)}{H(a_n, \epsilon_n)} = 0, \tag{9}$$

where

$$H(a_n, \epsilon_n) = \min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right) - ng(a_n),$$
$$G(a_n) = g(a_n + \frac{1}{g(a_n)}) - g(a_n),$$

where $F_{g_1}(a_n, \epsilon_n)$ *and* $F_{g_2}(a_n, \epsilon_n)$ *are defined as in Lemma 1.Then as* $n \to \infty$ *it holds* $P(I|C) \to 1$*.*

Example 3 *Let* $g(x) := x^\beta$*. For power functions,through Taylor expansion it holds*

$$g\left(a_n + \frac{1}{g(a_n)}\right) - g(a_n) = \frac{\beta}{a_n} + o\left(\frac{1}{a_n}\right) = o\left(\log g(a_n)\right)$$

hence condition (9) *holds as a consequence of* (8)*. If we assume that* $\epsilon_n = o(a_n)$*, by Taylor expansion we obtain*

$$\min\big(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\big) = na_n^\beta + C_\beta^2 \frac{n}{n-1} a_n^{\beta-2}\epsilon_n^2 + o(a_n^{\beta-2}\epsilon_n^2).$$

Condition (8) *then becomes*

$$\lim_{n\to\infty} \frac{n \log a_n}{a_n^{\beta-2}\epsilon_n^2} = 0.$$

Case 1: $1 < \beta \leq 2$*.*

To make (9) *hold, we need* ϵ_n *be large enough, specifically,*

$$a_n^{1-\frac{\beta}{2}} \sqrt{\log a_n} = o\left(\epsilon_n\right) = o\left(a_n\right)$$

which shows that $\epsilon_n \to \infty$*.*

Case 2: $\beta > 2$*.*

In this case, if we take $n = a_n^\alpha$ *with* $0 < \alpha < \beta - 2$*, then condition* (9) *holds for arbitrary sequences* ϵ_n *bounded by below away from* 0*. The sequence* ϵ_n *may also tend to* 0; *indeed with* $\epsilon_n = O(1/\log a_n)$*, condition* (9) *holds. Also setting* $a_n := n^\alpha$ *for* $\alpha > 0$ *there exist sequences* ϵ_n *which tend to* 0 *such that the conclusion in Theorem 2 holds.*

Example 4 Let $g(x) := e^x$*. Through Taylor expansion*

$$g\left(a_n + \frac{1}{g(a_n)}\right) - g(a_n) = 1 + o\left(\frac{1}{a_n}\right) = o\left(\log g(a_n)\right) = o\left(a_n\right),$$

and if $\epsilon_n \to 0$*, it holds*

$$\min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right) = ne^{a_n} + \frac{1}{2}\frac{n}{n-1}e^{a_n}\epsilon_n^2 + o(e^{a_n}\epsilon_n^2).$$

Hence condition (9) *follows from condition* (8)*; furthermore condition* (8) *follows from*

$$\lim_{n\to\infty} \frac{na_n}{e^{a_n}\epsilon_n^2} = 0$$

if we set $a_n := n^\alpha$ *where* $\alpha > 0$ *then condition* (9) *holds, and* ϵ_n *is rapidly decreasing to* 0*; indeed we may choose* $\epsilon_n = o(\exp(-a_n/4))$*.*

Corollary 5 *Let* $X_1, .., X_n$ *be independent r.v's with common Weibull density with shape parameter* k *and scale parameter* 1*,*

$$p(x) = \begin{cases} kx^{k-1}e^{-x^k} & \text{when } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $k > 2$*. Let*

$$a_n = n^{\frac{1}{\alpha}},$$

for some $0 < \alpha < k - 2$ *and let* ϵ_n *be a positive sequence such that*

$$\lim_{n\to\infty} \frac{n \log a_n}{a_n^{k-2}\epsilon_n^2} = 0.$$

Then

$$\lim_{n\to\infty} P(I|C) = 0.$$

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Proof: Set $g(x) = x^k - (k-1)\log x$, which is a convex function for $k > 2$. Also when $x \to \infty$, $g'(x)$ and $g''(x)$ are both infinitely small with respect to $g(x)$ as $x \to \infty$.

Both conditions (8) and (9) in Theorem 2 are satisfied. As regards to condition (9), notice firstly that, under the Weibull density by Taylor expansion

$$g(a_n + \epsilon_n) = g(a_n) + g'(a_n)\epsilon_n + \frac{1}{2}g''(a_n)\epsilon_n^2 + o\left(g''(a_n)\epsilon_n^2\right).$$

Hence it holds

$$\log g\left(a_n+\epsilon_n\right) \leq \log\left(3 g(a_n)\right) \leq \log\left(3 a_n^k\right) = \log 3 + k \log a_n.$$

Using Taylor expansion in $g(a_n+\epsilon_n)$ and $g\left(a_n - \frac{\epsilon_n}{n-1}\right)$, it holds

$$\begin{aligned}
F_{g_1}(a_n,\epsilon_n) - n g(a_n) &= g(a_n+\epsilon_n) + (n-1) g\left(a_n - \frac{\epsilon_n}{n-1}\right) - n g(a_n) \\
&= \left(g(a_n) + g'(a_n)\epsilon_n + \frac{1}{2} g''(a_n)\epsilon_n^2 + o\left(g''(a_n)\epsilon_n^2\right)\right) \\
&\quad + \left((n-1) g(a_n) - g'(a_n)\epsilon_n + \frac{1}{2} g''(a_n)\frac{\epsilon_n^2}{n-1} + o\left(g''(a_n)\epsilon_n^2\right)\right) - n g(a_n) \\
&\geq \frac{1}{2} g''(a_n)\epsilon_n^2 + o\left(g''(a_n)\epsilon_n^2\right) = \frac{k(k-1)}{2} a_n^{k-2}\epsilon_n^2 + o\left(a_n^{k-2}\epsilon_n^2\right).
\end{aligned}$$

In the same way, it holds when $a_n \to \infty$

$$F_{g_2}(a_n,\epsilon_n) - n g(a_n) \geq \frac{k(k-1)}{2} a_n^{k-2}\epsilon_n^2 + o\left(a_n^{k-2}\epsilon_n^2\right).$$

Thus we have

$$H(a_n,\epsilon_n) \geq \frac{k(k-1)}{2} a_n^{k-2}\epsilon_n^2 + o\left(a_n^{k-2}\epsilon_n^2\right).$$

Hence, when $n \to \infty$, with (??), (??), the condition (8) of Theorem (2) becomes

$$\begin{aligned}
\frac{n \log g\left(a_n+\epsilon_n\right)}{H(a_n,\epsilon_n)} &\leq \frac{n \log 3 + k n \log a_n}{\frac{k(k-1)}{2} a_n^{k-2}\epsilon_n^2 + o\left(a_n^{k-2}\epsilon_n^2\right)} \\
&\leq \frac{2 k n \log a_n}{\frac{k(k-1)}{4} a_n^{k-2}\epsilon_n^2} = \frac{8}{k-1}\frac{n \log a_n}{a_n^{k-2}\epsilon_n^2} \longrightarrow 0.
\end{aligned}$$

The last step holds from condition (??). As for condition (9) of Theorem (2), when $a_n \to \infty$, it holds

$$\begin{aligned}
n G(a_n) &= n g\left(a_n + \frac{1}{g(a_n)}\right) - n g(a_n) \\
&= n g(a_n) + n\frac{g'(a_n)}{g(a_n)} + o\left(\frac{g'(a_n)}{g(a_n)}\right) - n g(a_n) \\
&= n\frac{g'(a_n)}{g(a_n)} + o\left(\frac{g'(a_n)}{g(a_n)}\right) = o(n).
\end{aligned}$$

Hence under condition (??), it holds $nG(a_n) = o(H(a_n,\epsilon_n))$, which means that condition (9) of Theorem 2 holds under condition (??), which completes the proof.

4 Very Large Deviation for Exponential Density Functions associated to non-convex Functions

In this section, we pay attention to exponential density functions whose exponents are non-convex functions. Namely, i.i.d random variables $X_1, ..., X_n$ have common density with

$$f(x) = c \exp\Big(- (g(x) + q(x)) \Big)$$

assuming that the convex function g is twice differentiable and $q(x)$ is of smaller order than $\log g(x)$ for large x.

Theorem 6 *$X_1, ..., X_n$ are i.i.d. real valued random variables with common density $f(x) = c \exp\left(-(g(x) + q(x))\right)$, where $g(x)$ is some positive convex function on $\mathbb{R}^+$ and g is twice differentiable. Assume that on$[X, \infty)$, $g(x)$ is increasing on $[X, \infty)$ and satisfies*

$$\lim_{x \to \infty} g(x)/x = \infty.$$

Let $M(x)$ be some nonnegative continuous function on $\mathbb{R}^+$ for which

$$-M(x) \leq q(x) \leq M(x) \quad \textit{for all positive } x$$

together with

$$M(x) = O\left(\log g(x)\right) \tag{10}$$

as $x \to \infty$.

Let a_n be some positive sequence such that $a_n \to \infty$ and $\epsilon_n = o(a_n)$ be a positive sequence. Assume

$$\liminf_{n \to \infty} \frac{\log g(a_n)}{\log n} > 0 \tag{11}$$

$$\lim_{n \to \infty} \frac{n \log g\left(a_n + \epsilon_n\right)}{H(a_n, \epsilon_n)} = 0, \tag{12}$$

$$\lim_{n \to \infty} \frac{nG(a_n)}{H(a_n, \epsilon_n)} = 0, \tag{13}$$

where

$$H(a_n, \epsilon_n) = \min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right) - ng(a_n),$$

$$G(a_n) = g\left(a_n + \frac{1}{g(a_n)}\right) - g(a_n),$$

where $F_{g_1}(a_n, \epsilon_n)$ and $F_{g_2}(a_n, \epsilon_n)$ are defined as in Lemma 1.

Then it holds

$$P(I|C) \to 1 \quad \textit{when } n \to \infty.$$

We now provide examples of densities which define r.v's X_i''s for which the above Theorem 6 applies. These densities appear in a number of questions pertaining to uniformity in large deviation approximations; see [15] Ch 6.

Example 7 ***Almost Log-concave densities****: p can be written as*

$$p(x) = c(x) \exp -h(x), \qquad x < \infty$$

with h a convex function, and where for some $x_0 > 0$ and constants $0 < c_1 < c_2 < \infty$, we have

$$c_1 < c(x) < c_2 \textit{ for } x_0 < x < \infty.$$

Densities which satisfy the above condition include the Normal, the Gamma, the hyperbolic density, etc.

Example 8 ***Gamma-like densities*** *are defined through densities of the form*

$$p(x) = c(x) \exp -h(x)$$

for all $x > 0$, with $0 < c_1 < c(x) < c_2 \leq \infty$ when x is larger than some $x_0 > 0$ and $h(x)$ is a convex function which satisfies $h(x) = \tau + h_1(x)$ with, for $x_1 < x_2$,

$$a_1 \log \frac{x_2}{x_1} - b_1 < h_1(x_2) - h_1(x_1) < a_2 \log \frac{x_2}{x_1} - b_2$$

where a_1, a_2, b_1 and b_2 are positive constants with $a_2 < 1$.

A wide class of densities for which our results apply is when there exist constants $x_0 > 0$, $\alpha > 0$, $\tau > 0$ and A such that

$$p(x) = Ax^{\alpha-1} l(x) \exp(-\tau x) \qquad x > x_0$$

where $l(x)$ is slowly varying at infinity.

Example 9 ***Almost Log-concave densities 1****: p can be written as*

$$p(x) = c(x) \exp -g(x), \qquad 0 < x < \infty$$

with g a convex function, and where for some $x_0 > 0$ and constants $0 < c_1 < c_2 < \infty$, we have

$$c_1 < c(x) < c_2 \textit{ for } x_0 < x < \infty,$$

and $g(x)$ is increasing on some interval $[X, \infty)$ and satisfies

$$\lim_{x \to \infty} g(x)/x = \infty.$$

Examples of densities which satisfy the above conditions include the Normal, the hyperbolic density, etc.

Example 10 ***Almost Log-concave densities 2:*** *A wide class of densities for which our results apply is when there exist constants* $x_0 > 0$*,* $\alpha > 0$*, and* A *such that*

$$p(x) = Ax^{\alpha-1} l(x) \exp(-g(x)) \qquad x > x_0$$

where $l(x)$ *is slowly varying at infinity,* g *a convex function, increasing on some interval* $[X, \infty)$ *and satisfies*

$$\lim_{x \to \infty} g(x)/x = \infty.$$

Remark 11 *All density functions in Examples* (9) (10) *satisfy the assumptions of the above Theorem 6 . Also the conditions in Theorem 6 about* a_n *and* ϵ_n *are the same as those in the convex case, so that if* $g(x)$ *is some power function with index larger than* 2*,* ϵ_n *can go to* 0 *more rapidly than* $O(1/\log a_n)$ *(see Example* 3*); If* $g(x)$ *is of exponential function form,* ϵ_n *goes to* 0 *more rapidly than any power* $1/a_n$ *(see Example* 4 *).**

5 Application

An extended LDP holds for the partial sum S_1^n where the i.i.d. summands X_i's are unbounded above whenever

$$\lim_{n \to \infty} -\frac{\log P(S_1^n/n > x_n)}{I(x_n)} = 1$$

holds where $\lim_{n\to\infty} x_n = +\infty$. In the above display the Cramer function $I(x)$ is defined for all $x > EX$ through

$$I(x) := \sup_t tx - \log E \exp tX.$$

Thne following result holds (see [8], Proposition 1.1). Assume that X is unbounded above and satisfies the Cramer condition. Assume further that

$$-\log P(X > x) = I(x)(1 + o(1))) \tag{14}$$

as $x \to \infty$. Then for any sequence a_n going to infinity with n it holds

$$-\log P(S_1^n/n > a_n) = nI(a_n)(1 + o(1))) \tag{15}$$

as $n \to \infty$. It is readily seen that (14) holds in any of the cases considered in the present paper (see [8], Remark 1.1). See also [4] for a sharp result.

We now consider the local behaviour of a random walk with independent summands X_i, $1 \leq i \leq n$ which are identically distributed as

X. Let $a > EX$. We consider random paths $T_n := (S_1^1, S_1^2, .., S_1^n)$ which satisfy $(S_1^n > na)$ hence under a *large deviation condition* pertaining to the end value. In the following result we state that the trajectory T_n exhibits a peculiar feature.

Let $k = k_n$ be an integer sequence such that $\lim_{n\to\infty} k = \infty$ together with $\lim_{n\to\infty} k/n = 0$, and $\alpha_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{na - k\alpha_k}{n-k} = \infty.$$

Denote A_k the event

$$A_k := (\text{there exists } j,\ 1 \leq j \leq n-k \text{ such that } \Delta_{j,k} > \alpha_k).$$

It holds

Proposition 12 *When X satisfies the hypotheses in Theorem 6 it holds*

$$P\left(A_k | \, S_1^n > na\right) \to 1.$$

Proof. The proof is simple and we briefly sketch the argument. Clearly

$$\begin{aligned} P\left(A_k | \, S_1^n > na\right) &= 1 - P\left(\cap_{j=0}^{[n/k]} \left(S_{kj+1}^{kj} < k\alpha_k \right) \Big| \, S_1^n > na \right) \\ &\leq 1 - \left(P\left(S_1^k < k\alpha_k \big| \, S_1^n > na\right)\right)^{[n/k]} =: 1 - P. \end{aligned}$$

Now applying Bayes Theorem and the independence of the r.v's X_i's, it holds

$$P \leq \frac{P\left(S_{k+1}^n > \frac{na - k\alpha_k}{n-k}\right)}{P\left(S_1^n > na\right)}.$$

Under the present hypotheses (14) holds.Using (15) in the numerator and the classical first order LDP result

$$\log P\left(S_1^n > na\right) = -nI(a)\left(1 + o(1)\right)$$

in the denominator, it follows that $P \to 0$ as $n \to \infty$, which concludes the proof. ■

The consequence of Theorem 6 is that on this segment of length k where the slope exceeds α_k all the summands are of order α_k so that the behaviour of the trajectory is nearly linear. Numerical evidence confirm the theoretical ones; for very large a and fixed (large) n , not surprisingly, the *democratic localisation* holds on the entire trajectory , in accordance with the results in this paper; therefore T_n is nearly a straight line from the origin up to the point (n, na_n). For smaller values of a (typically for a defined through $P\left(S_1^n > na\right)$ of order 10^{-3} the phenomenon quoted in the above proposition holds: T_n consists in a number of oblic segments. When n is allowed to increase, the segments are longer and longer, with increasing slope.

6 Appendix

6.1 Proof of Lemma 1

Write $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}_+^n$, we firstly define the following sets. Let for all k between O and n

$$A_k := \left\{\text{there exist } i_1, .., i_k \text{ such that } x_{i_j} \geq a_n + \epsilon_n \text{ for all } j \text{ with } 1 \leq j \leq k\right\}$$

and

$$B_k := \left\{\text{there exist } i_1, .., i_k \text{ such that } x_{i_j} \leq a_n - \epsilon_n \text{ for all } j \text{ with } 1 \leq j \leq k\right\}.$$

Define

$$A = \bigcup \lim_{k=1}^{n} A_k$$

and

$$B = \bigcup \lim_{k=1}^{n} B_k.$$

It then holds

$$I^c = A \cup B.$$

It follows that

$$\begin{aligned} I_g(I^c \cap C) &= I_g\left((A \cup B) \cap C\right) = \inf_{\mathbf{x} \in (A \cap C) \cup (B \cap C)} I_g(\mathbf{x}) \\ &= \min\left(I_g(A \cap C), I_g(B \cap C)\right). \end{aligned}$$

Thus we may calculate the minimum values of both $I_g(A \cap C)$ and $I_g(B \cap C)$ respectively, and finally $I_g(I^c \cap C)$.

Step 1: In this step we prove that

$$I_g(A \cap C) = F_{g_1}(a_n, \epsilon_n). \tag{16}$$

Let $\mathbf{x} := (x_1, ..., x_n)$ belong to $A \cap C$ and assume that $I_g(A \cap C) = I_g(\mathbf{x})$. Without loss of generality, assume that the x_i's are ordered ascendently, $x_1 \leq ... \leq x_i \leq x_{i+1 \leq}, ... \leq x_n$ and let i and $k := n - i$ with $1 \leq i \leq n$ such that

$$\overbrace{x_1 \leq ... \leq x_i}^{n-k} < a_n + \epsilon_n \leq \overbrace{x_{i+1} \leq ... \leq x_n}^{k}.$$

We first claim that $k < n$. Indeed let $\mathbf{y} := (y_1 = a_n - \epsilon_n, y_2 = ... = y_{n-1} = a_n + \epsilon_n)$ which clearly belongs to $A \cap C$. For this $\mathbf{y}$ it holds $I_g(\mathbf{y}) = (n-1)g(a_n + \epsilon_n) + g(a_n - \epsilon_n)$ which is strictly smaller than $ng(a_n + \epsilon_n) = I_g(A_n \cap C)$ for large n. We have proved that $\mathbf{x}$ does not belong to $A_n \cap C$.

Let $\alpha_{i+1}, ..., \alpha_n$ be nonnegative, and write $x_{i+1}, ..., x_n$ as

$$x_{i+1} = a_n + \epsilon_n + \alpha_{i+1}, ..., x_n = a_n + \epsilon_n + \alpha_n.$$

Under condition (C), it holds

$$\begin{aligned} x_1 + ... + x_i &\geq na_n - (x_{i+1} + ... + x_n) \\ &= na_n - k(a_n + \epsilon_n) - (\alpha_{i+1} + ... + \alpha_n) \,. \end{aligned}$$

Applying Jensen's inequality to the convex function g, we have

$$\begin{aligned} \sum_{i=1}^{n} g(x_i) &= (g(x_{i+1}) + ... + g(x_n)) + (g(x_1) + ... + g(x_i)) \\ &\geq (g(x_{i+1}) + ... + g(x_n)) + (n-k)g(x^*), \end{aligned}$$

where equality holds when $x_1 = ... = x_i = x^*$, with

$$x^* = \frac{na_n - k(a_n + \epsilon_n) - (\alpha_{i+1} + ... + \alpha_n)}{n-k}.$$

Define now the function function $(\alpha_{i+1}, ..., \alpha_n, k) \rightarrow f(\alpha_{i+1}, ..., \alpha_n, k)$ through

$$\begin{aligned} f(\alpha_{i+1}, ..., \alpha_n, k) &= g(x_{i+1}) + ... + g(x_n) + (n-k)g(x^*) \\ &= g(a_n + \epsilon_n + \alpha_{i+1}) + ... + g(a_n + \epsilon_n + \alpha_n) + (n-k)g(x^*). \end{aligned}$$

Then $I_g(A \cap C)$ is given by

$$I_g(A \cap C) = \inf_{\alpha_{i+1}, ..., \alpha_n \geq 0, 1 \leq k \leq n} f(\alpha_{i+1}, ..., \alpha_n, k).$$

We now obtain (16) through the properties of the function f. Using (**??**), the first order partial derivative of $f(\alpha_{i+1}, ..., \alpha_n, k)$ with respect to α_{i+1} is

$$\frac{\partial f(\alpha_{i+1}, ..., \alpha_n, k)}{\partial \alpha_{i+1}} = g'(a_n + \epsilon_n + \alpha_{i+1}) - g'(x^*) > 0,$$

where the inequality holds since $g(x)$ is strictly convex and $a_n + \epsilon_n + \alpha_{i+1} > x^*$. Hence $f(\alpha_{i+1}, ..., \alpha_n, k)$ is an increasing function with respect to α_{i+1}. This implies that the minimum value of f is attained when $\alpha_{i+1} = 0$. In the same way, we have $\alpha_{i+1} = ... = \alpha_n = 0$. Therefore it holds

$$I_g(A \cap C) = \inf_{1 \leq k \leq n} f(\mathbf{0}, k),$$

with

$$f(\mathbf{0}, k) = kg(a_n + \epsilon_n) + (n-k)g(x_0^*),$$

where

$$x_0^* = a_n - \frac{k}{n-k}\epsilon_n.$$

The function $y \to f(\mathbf{0}, y)$ with $0 < y < n$ is increasing with respect to y, since

$$\begin{aligned}\frac{\partial f(\mathbf{0}, y)}{\partial y} &= g(a_n + \epsilon_n) - g(x_0^*) - \frac{n\epsilon_n}{n-y} g'(x_0^*) \\ &= \frac{n\epsilon_n}{n-y} \left(\frac{g(a_n + \epsilon_n) - g(x_0^*)}{a_n + \epsilon_n - x_0^*} - g'(x_0^*) \right) > 0,\end{aligned}$$

due to the convexity of $g(x)$ and $a_n + \epsilon_n > x_0^*$. Hence $f(\mathbf{0}, k)$ is increasing with respect to k; the minimal value of $f(\mathbf{0}, k)$attains with $k = 1$. Thus we have

$$I_g(A \cap C) = f(\mathbf{0}, 1) = F_{g_1}(a_n, \epsilon_n)$$

which proves (16).

Step 2: In this step, we follow the same proof as above and prove that

$$I_g(B \cap C) = F_{g_2}(a_n, \epsilon_n).$$

With $\mathbf{x}$ defined through $I_g(\mathbf{x}) := I_g(B \cap C)$ with the coordinates of $\mathbf{x}$ ranked in ascending order, with j such that $1 \le j \le n$ and

$$\overbrace{x_1 \le ... \le x_j}^{j} < a_n + \epsilon_n \le \overbrace{x_{j+1} \le ... \le x_n}^{n-j}$$

we obtain $j < n$ through the same argument as above. Denote $x_1, ..., x_j$ by

$$x_1 = a_n - \epsilon_n - \alpha_1, ..., x_n = a_n - \epsilon_n - \alpha_j,$$

where $\alpha_1, ..., \alpha_j$ are nonnegative. Under condition (C), it holds

$$\begin{aligned}x_{j+1} + ... + x_n &\ge na_n - (x_1 + ... + x_j) \\ &= na_n - j(a_n - \epsilon_n) + (\alpha_1 + ... + \alpha_j).\end{aligned}$$

Using Jensen's inequality to the convex function $g(x)$, we have

$$\begin{aligned}\sum_{i=1}^{n} g(x_i) &= (g(x_1) + ... + g(x_j)) + (g(x_{j+1}) + ... + g(x_n)) \\ &\ge (g(x_1) + ... + g(x_j)) + (n-j)g(x^\sharp),\end{aligned}$$

where the equality holds when $x_{j+1} = ... = x_n = x^\sharp$, with

$$x^\sharp = \frac{na_n - j(a_n - \epsilon_n) + (\alpha_1 + ... + \alpha_j)}{n-j}.$$

Define the function $(\alpha_{i+1}, ..., \alpha_n, k) \to f(\alpha_{i+1}, ..., \alpha_n, k)$ through

$$\begin{aligned} f(\alpha_1, ..., \alpha_j, j) &= g(x_1) + ... + g(x_j) + (n-j)g(x^\sharp) \\ &= g(a_n - \epsilon_n - \alpha_1) + ... + g(a_n - \epsilon_n - \alpha_j) + (n-j)g(x^\sharp), \end{aligned}$$

then $I_g(A \cap C)$ is given by

$$I_g(A \cap C) = \inf_{\alpha_1, ..., \alpha_j \geq 0, 1 \leq j \leq n} f(\alpha_1, ..., \alpha_j, j).$$

Using (**??**), the first order partial derivative of $f(\alpha_1, ..., \alpha_j, j)$ with respect to α_1 is

$$\frac{\partial f(\alpha_1, ..., \alpha_j, j)}{\partial \alpha_1} = -g'(a_n - \epsilon_n - \alpha_1) + g'(x^\sharp) > 0,$$

where the inequality holds since $g(x)$ is convex and $a_n - \epsilon_n - \alpha_1 < x^\sharp$. Hence $f(\alpha_1, ..., \alpha_j, j)$ is increasing with respect to α_1. This yields

$$\alpha_1 = ... = \alpha_j = 0.$$

Therefore it holds

$$I_g(B \cap C) = \inf_{1 \leq k \leq n} f(\mathbf{0}, j),$$

with

$$f(\mathbf{0}, j) = jg(a_n - \epsilon_n) + (n-j)g(x_0^\sharp),$$

where

$$x_0^\sharp = a_n + \frac{j}{n-j}\epsilon_n.$$

The function $y \to f(\mathbf{0}, y)$ with $0 < y < n$ is increasing with respect to y, since

$$\begin{aligned} \frac{\partial f(\mathbf{0}, y)}{\partial y} &= g(a_n - \epsilon_n) - g(x_0^\sharp) + \frac{n\epsilon_n}{n-j} g'(x_0^\sharp) \\ &= \frac{n\epsilon_n}{n-y} \left(g'(x_0^\sharp) - \frac{g(x_0^\sharp) - g(a_n - \epsilon_n)}{x_0^\sharp - (a_n - \epsilon_n)} \right) > 0, \end{aligned}$$

by is convexity of g ; in the above display $x_0^\sharp > a_n - \epsilon_n$. Hence $f(\mathbf{0}, k)$ is increasing with respect to k. Thus we have

$$I_g(B \cap C) = f(\mathbf{0}, 1) = F_g(a_n, \epsilon_n)$$

which proves the claim.

Thus the proof is completed using (16) and (**??**).

6.2 Proof of Theorem 2

For $x = (x_1, ..., x_n) \in \mathbb{R}_+^n$, define

$$S_g(r) = \left\{ x : \sum_{1 \leq i \leq n} g(x_i) \leq r \right\}.$$

Then for any Borel set A in $\mathbb{R}^n$ it holds

$$\begin{aligned} P(A) &= \int_A \exp\left(- \sum_{1 \leq i \leq n} p(x_i) \right) dx_1, ..., dx_n \\ &= \exp(-I_g(A)) \int_A dx_1, ..., dx_n \int 1_{\left[\sum_{1 \leq i \leq n} g(x_i) - I_g(A), \infty\right)}(s) e^{-s} ds \\ &= \exp(-I_g(A)) \int_0^\infty Volume(A \cap S_g(I_g(A) + s)) e^{-s} ds. \end{aligned}$$

The proof is divided in three steps.

Step 1: We prove that

$$P(C) \geq c^n \exp\left(-I_g(C) - \tau_n - n \log g(a_n)\right). \tag{17}$$

where

$$\tau_n = ng\left(a_n + \frac{1}{g(a_n)}\right) - ng(a_n). \tag{18}$$

By convexity of the function g, and using condition (C), applying Jensen's inequality, with $x_1 = ... = x_n = a_n$ it holds

$$I_g(C) = ng(a_n).$$

We now consider the largest lower bound for

$$\log Volume\left(C \cap S_g(I_g(C) + \tau_n)\right).$$

Denote $B = \left\{ \mathbf{x} : x_i \in [a_n, a_n + \frac{1}{g(a_n)}] \right\}$, $S_g(I_g(C)+\tau_n) = \{\mathbf{x} : \sum_{i=1}^n g(x_i) \leq ng(a_n) + \tau_n\}$.

For large n and any $\mathbf{x} := (x_1, .., x_n)$ in B, it holds

$$\sum_{i=1}^n g(x_i) \leq \sum_{i=1}^n g\left(a_n + \frac{1}{g(a_n)}\right) = ng\left(a_n + \frac{1}{g(a_n)}\right) = ng(a_n) + \tau_n,$$

where we used the fact that g is an increasing function for large argument. Hence

$$B \subset S_g(I_g(C) + \tau_n).$$

It follows that

$$\log Volume\left(C \cap S_g(I_g(C)+\tau_n)\right) \geq \log Volume(B) = \log\left(\frac{1}{g(a_n)}\right)^n = -n\log g(a_n) \tag{19}$$

which in turn using (**??**) and (19),implies

$$\begin{aligned}\log P(C) &:= \log \int_C \exp\left(-\sum_{1\leq i\leq n} g(x_i)\right) dx_1,...,dx_n \\ &\geq \log\left(\exp(-I_g(C))\int_{\tau_n}^{\infty} Volume(C\cap S_g(I_g(C)+s))e^{-s}ds\right) \\ &\geq -I_g(C) - \tau_n + \log Volume(C\cap S_g(I_g(C)+\tau_n)) \\ &\geq -I_g(C) - \tau_n - n\log g(a_n),\end{aligned}$$

This proves the claim.

Step 2: In this step, we prove that

$$P(I^c\cap C) \leq c^n \exp\left(-I_g(I^c\cap C) + n\log I_g(I^c\cap C) + \log(n+1)\right). \tag{20}$$

For any Borel set A in $\mathbb{R}^n$ it holds , for positive s, let

$$S_g(I_g(A)+s) = \left\{\mathbf{x} : \sum_{1\leq i\leq n} g(x_i) \leq I_g(A)+s\right\}$$

and

$$F = \left\{\mathbf{x} : g(x_i) \leq I_g(A)+s, i=1,...,n\right\}.$$

It holds.

$$S_g(I_g(A)+s) \subset F.$$

Since $\lim_{x\to\infty} g(x)/x = +\infty$

$$F \subset \{\mathbf{x} : x_i \leq (I_g(A)+s), i=1,...,n\},$$

which yields

$$S_g(I_g(A)+s) \subset \{\mathbf{x} : x_i \leq (I_g(A)+s), i=1,...,n\},$$

from which we obtain

$$Volume(A\cap S_g(I_g(A)+s)) \leq Volume(S_g(I_g(A)+s)) \leq (I_g(A)+s)^n.$$

With this inequality, the upper bound of integration (**??**) can be given when $a_n \to \infty$.

$$\begin{aligned}\log P(A) = & \log \int_A \exp\left(- \sum_{1\le i\le n} g(x_i) \right) dx_1, ..., dx_n \\ & = -I_g(A) + \log \int_0^\infty Volume(A \cap S_g(I_g(A)+s))e^{-s} ds \\ & \le -I_g(A) + \log \int_0^\infty \left(I_g(A)+s\right)^n e^{-s} ds,\end{aligned}$$

with integrating repeatedly by parts it holds

$$\begin{aligned}& \int_0^\infty \left(I_g(A)+s\right)^n e^{-s} ds \\ & = I_g(A)^n + n \int_0^\infty \left(I_g(A)+s\right)^{n-1} e^{-s} ds \\ & = I_g(A)^n + nI_g(A)^{n-1} + n(n-1) \int_0^\infty \left(I_g(A)+s\right)^{n-2} e^{-s} ds \\ & \le (n+1)I_g(A)^n,\end{aligned} \tag{21}$$

hence we have

$$\begin{aligned}& \log \int_A \exp\left(- \sum_{1\le i\le n} g(x_i) \right) dx_1, ..., dx_n \\ & \le -I_g(A) + \log\left((n+1)I_g(A)^n\right) \\ & = -I_g(A) + n \log I_g(A) + \log(n+1).\end{aligned}$$

Replace A by $I^c \cap C$. We then obtain

$$P(I^c \cap C) \le c^n \exp\left(-I_g(I^c \cap C) + n \log I_g(I^c \cap C) + \log(n+1)\right)$$

as sought.

Step 3: In this step, we will complete the proof , showing that

$$\lim_{a_n \to \infty} \frac{P(I^c \cap C)}{P(C)} = 0.$$

By Lemma 1,

$$I_g(I^c \cap C) = \min\left(F_{g_1}(a_n, \epsilon_n), F_{g2}(a_n, \epsilon_n)\right).$$

Using (17) and (20) it holds

$$\frac{P(I^c \cap C)}{P(C)} \le \exp\left(-H(a_n, \epsilon_n) + n \log I_g(I^c \cap C) + \tau_n + n \log g(a_n) + \log(n+1)\right).$$

Under conditions (9), by (18) when $a_n \to \infty$, we have

$$\frac{\tau_n}{H(a_n, \epsilon_n)} = \frac{nG(a_n)}{H(a_n, \epsilon_n)} \longrightarrow 0,$$

Using conditions (**??**) and (8), when $a_n \to \infty$,

$$\frac{n \log g(a_n)}{H(a_n, \epsilon_n)} \longrightarrow 0, \qquad \textit{and} \qquad \frac{\log(n+1)}{H(a_n, \epsilon_n)} \longrightarrow 0.$$

As to the term $n \log I_g(I^c \cap C)$, we have

$$\begin{aligned} n \log I_g(I^c \cap C) &= n \min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right) \\ &\leq n \log\left(ng(a_n + \epsilon_n)\right) \\ &= n \log n + n \log g\left(a_n + \epsilon_n\right). \end{aligned}$$

Under condition (8), when $a_n \to \infty$, $n \log g\left(a_n + \epsilon_n\right)$ is of small order with respect to $H(a_n, \epsilon_n)$ as n tends to infinity. Under condition (**??**), for a_n large enough, there exists some positive constant Q such that $\log n \leq Q \log g(a_n)$. Hence we have

$$n \log n \leq Qn \log g(a_n)$$

which under condition (8), yields that $n \log n$ is negligible with respect to $H(a_n, \epsilon_n)$. Hence when $a_n \to \infty$, it holds

$$\frac{n \log\left(I_g(I^c \cap C)\right)}{H(a_n, \epsilon_n)} \longrightarrow 0.$$

Further, (**??**), (**??**) and (**??**) make (**??**) hold. This completes the proof.

6.3 Proof of Theorem 6

The proof is is the same vein as that of Theorem 2; some care has to be taken in order to get similar bounds as developped in the convex case.

Denote $\mathbf{x} = (x_1, ..., x_n)$ in $\mathbb{R}^n$ and, for a Borel set $A \in {\mathbb{R}_+}^n$ define

$$I_{g,q}(A) = \inf_{\mathbf{x} \in A} I_{g,q}(\mathbf{x}),$$

where

$$I_{g,q}(\mathbf{x}) := \sum_{1 \leq i \leq n} \left(g(x_i) + q(x_i)\right).$$

Also for any positive r define

$$S_{g,q}(r) = \left\{ \mathbf{x} : \sum_{1 \leq i \leq n} \left(g(x_i) + q(x_i)\right) \leq r \right\}.$$

Then it holds

$$
\begin{aligned}
P(A) &= \int_A \exp\left(-\sum_{1\le i\le n}(g(x_i)+q(x_i))\right) dx_1, ..., dx_n \\
&= \exp(-I_{g,q}(A)) \int_A dx_1, ..., dx_n \int 1_{\left[\sum_{1\le i\le n}(g(x_i)+q(x_i))-I_{g,q}(A),\infty\right)}(s)e^{-s}ds \\
&= \exp(-I_{g,q}(A)) \int_0^\infty Volume(A\cap S_{g,q}(I_{g,q}(A)+s))e^{-s}ds. \qquad (22)
\end{aligned}
$$

Step 1: In this step we prove that

$$I_{g,q}(C) \ge I_{g_1}(C) \ge nh(a_n) = ng(a_n) - nN\log g(a_n).$$

For large x it holds

$$g(x) - M(x) \le g(x) + q(x) \le g(x) + M(x). \qquad (23)$$

Set $g_1(x) = g(x) - M(x)$ and $g_2(x) = g(x) + M(x)$, then it follows

$$I_{g_1}(C) \le I_{g,q}(C) \le I_{g_2}(C). \qquad (24)$$

In the same way, it holds

$$I_{g_1}(I^c\cap C) \le I_{g,q}(I^c\cap C) \le I_{g_2}(I^c\cap C). \qquad (25)$$

By condition (10), there exists some sufficiently large positive y_0 and some positive constant N such that for $x\in[y_0,\infty)$

$$M(x) \le N\log g(x). \qquad (26)$$

Set $r(x) = g(x) - N\log g(x)$, the second order derivative of $r(x)$ is

$$r''(x) = g''(x)\left(1-\frac{N}{g(x)}\right) + \frac{N\left(g'(x)\right)^2}{g^2(x)},$$

where the second term is positive. The function g is increasing on some interval $[X,\infty)$ where we also have $g(x) > x$. Hence there exists some $y_1\in[X,\infty)$ such that s $g(x) > N$ when $x\in[y_1,\infty)$. This implies that $r''(x) > 0$ and $r'(x) > 0$ and therefore $r(x)$ is convex and increasing on $[y_1,\infty)$.

In addition, $M(x)$ is bounded on any finite interval; there exists some $y_2\in[y_1,\infty)$ such that for all $x\in(0,y_2)$

$$M(x) \le N\log g(y_2). \qquad (27)$$

The function g is convex and increasing on $[y_2, \infty)$. Thus there exists y_3 such that

$$g'(y_3) > 2g'(y_2) \quad and \quad g(y_3) > 2N. \tag{28}$$

We now construct a function h as follows. Let

$$h(x) = r(x)\mathbf{1}_{[y_3,\infty)}(x) + s(x)\mathbf{1}_{(0,y_3)}(x), \tag{29}$$

where $s(x)$ is defined by

$$s(x) = r(y_3) + r'(y_3)(x - y_3). \tag{30}$$

We will show that

$$g_1(x) \geq h(x) \tag{31}$$

for $x \in (0, \infty)$.

If $x \in [y_3, \infty)$, then by (26), it holds

$$h(x) = r(x) = g(x) - N \log g(x) \leq g(x) - M(x) = g_1(x). \tag{32}$$

If $x \in (y_2, y_3)$, using (30), we have

$$s(x) \leq r(x) = g(x) - N \log g(x) \leq g(x) - M(x) = g_1(x), \tag{33}$$

where the first inequality comes from the convexity of $r(x)$. We now show that (31) holds when $x \in (0, y_2]$ if y_3 is large enough. For this purpose, set

$$t(x) = g(x) - s(x) - N \log g(y_2).$$

Take the first order derivative of t and use the convexity of g on $(0, y_2]$. We have

$$\begin{aligned}
t'(x) &= g'(x) - s'(x) = g'(x) - r'(y_3) = g'(x) - \left(g'(y_3) - \frac{Ng'(y_3)}{g(y_3)}\right) \\
&= g'(x) - \left(1 - \frac{N}{g(y_3)}\right) g'(y_3) \leq g'(y_2) - \left(1 - \frac{N}{g(y_3)}\right) g'(y_3) \\
&< \frac{1}{2}g'(y_3) - \left(1 - \frac{N}{g(y_3)}\right) g'(y_3) < 0,
\end{aligned}$$

where the inequalities in the last line hold from (28). Therefore t is decreasing on $(0, y_2]$. It follows that

$$t(x) \geq t(y_2) = g(y_2) - N \log g(y_2) - s(y_2) \geq g(y_2) - N \log g(y_2) - r(y_2) = 0,$$

which, together with (27), yields, when $x \in (0, y_2]$

$$g_1(x) = g(x) - M(x) \geq g(x) - N \log g(y_2) \geq s(x).$$

Together with (32), (33), this last display means that (31) holds.

We now prove that h is a convex function on on $(0,\infty)$.; indeed for x such that $0 < x \leq y_3$, $h''(x) = 0$, and if $x > y_3$, $h''(x) = r''(x) > 0$. The left derivative of $h(x)$ at y_3 is $h'(y_3^-) = r'(y_3)$, and it is obvious that the right derivative of $h(x)$ at y_3 is also $h'(y_3^+) = r'(y_3)$; hence h is derivable at y_3 and $h'(y_3) = r'(y_3)$, hence $h''(y_3) = r''(y_3) > 0$. This shows that h is convex on $(0,\infty)$.

Now under condition (C), using the convexity of h and (31), it holds

$$I_{g_1}(\mathbf{x}) = \sum_{i=1}^{n} (g(x_i) - M(x_i)) \geq \sum_{i=1}^{n} h(x_i) \geq nh\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = nh(a_n).$$

Using (24), we obtain the lower bound of $I_{g,q}(C)$ under condition (C) for a_n large enough (say, $a_n > y_3$)

$$I_{g,q}(C) \geq I_{g_1}(C) \geq nh(a_n) = nr(a_n) = ng(a_n) - nN \log g(a_n). \tag{34}$$

Step 2: In this step, we will show that the following lower bound of $P(C)$ holds

$$P(C) \geq c^n \exp\left(-I_{g,q}(C) - \tau_n - n \log g(a_n)\right), \tag{35}$$

where τ_n is defined by

$$\begin{aligned} \tau_n &= ng\left(a_n + \frac{1}{g(a_n)}\right) - ng(a_n) + nN \log g\left(a_n + \frac{1}{g(a_n)}\right) + nN \log g(a_n) \\ &= nG(a_n) + nN \log g(a_n) + nN \log g\left(a_n + \frac{1}{g(a_n)}\right). \end{aligned} \tag{36}$$

Denote $B = \left\{\mathbf{x} : x_i \in [a_n, a_n + \frac{1}{g(a_n)}].\right\}$. If $\mathbf{x} \in B$, by (26), which holds for large n (say, $a_n > y_3$ and assuming that g is an increasing function on (y_3,∞)), we have

$$\begin{aligned} I_{g,q}(\mathbf{x}) &\leq \sum_{i=1}^{n} (g(x_i) + M(x_i)) \leq \sum_{i=1}^{n} (g(x_i) + N \log g(x_i)) \\ &\leq \sum_{i=1}^{n} \left(g\left(a_n + \frac{1}{g(a_n)}\right) + N \log g\left(a_n + \frac{1}{g(a_n)}\right)\right) \\ &= ng\left(a_n + \frac{1}{g(a_n)}\right) + nN \log g\left(a_n + \frac{1}{g(a_n)}\right) \\ &= \tau_n + ng(a_n) - nN \log g(a_n) \leq \tau_n + I_{g,q}(C), \end{aligned}$$

where the last inequality holds from (34). Since $B \subset C$, we have

$$B \subset C \cap S_{g,q}(I_{g,q}(C) + \tau_n).$$

Now we may obtain the lower bound

$$\log Volume\left(C \cap S_{g,q}(I_{g,q}(C) + \tau_n)\right) \geq \log Volume(B) = -n \log g(a_n). \tag{37}$$

Using (22) and (37), it holds

$$\begin{aligned}
&\log \int_C \exp\left(-\sum_{1\leq i\leq n}(g(x_i)+q(x_i))\right)dx_1,...,dx_n \\
&= -I_{g,q}(C) + \log \int_0^{\infty} Volume(C \cap S_{g,q}(I_{g,q}(C)+s))e^{-s}ds \\
&\geq -I_{g,q}(C) + \log \int_{\tau_n}^{\infty} Volume(C \cap S_{g,q}(I_{g,q}(C)+\tau_n))e^{-s}ds \\
&\geq -I_{g,q}(C) - \tau_n - n\log g(a_n),
\end{aligned}$$

so (35) holds.

Step 3: We prove that

$$P(I^c \cap C) \leq c^n \exp\left(-I_{g,q}(I^c \cap C) + n \log I_g(I^c \cap C) + \log(n+1) + n\log 2\right). \tag{38}$$

For any Borel set A in $\mathbb{R}^n$ and any positive s,

$$S_{g,q}(I_{g,q}(A)+s) = \left\{\mathbf{x} : \sum_{1\leq i\leq n}(g(x_i)+q(x_i)) \leq I_{g,q}(A)+s\right\}$$

is included in $\{\mathbf{x} : g(x_i)+q(x_i) \leq I_{g,q}(A)+s, i = 1,...,n\}$ which in turn is included in $F = \{\mathbf{x} : g(x_i) - M(x_i) \leq (I_{g,q}(A)+s), i = 1,...,n\}$ by (23).

Set $H = \{\mathbf{x} := (x_1,..,x_n) : x_i \leq 2(I_{g,q}(A)+s), i = 1,...,n\}$, we will show it holds for a_n large enough

$$F \subset H.$$

Suppose that for some $\mathbf{x} := (x_1,..,x_n)$ in F ,some x_i is larger than $2(I_{g,q}(A)+s)$. For a_n large enough, by (34), it holds

$$\begin{aligned}
x_i &\geq 2(I_{g,q}(A)+s) \geq 2\left(ng(a_n) - nN\log g(a_n)\right) \\
&> 2\left(ng(a_n) - \frac{1}{4}ng(a_n)\right) = \frac{3}{2}ng(a_n).
\end{aligned}$$

Since $\frac{3}{2}ng(a_n) \geq \frac{3}{2}na_n$ for large n, by (26) and since $x \to g(x) - N\log g(x)$ is increasing, we have

$$
\begin{aligned}
g(x_i) - M(x_i) \geq g(x_i) - N\log g(x_i) &\geq g\left(2(I_{g,q}(A)+s)\right) - N\log g\left(2(I_{g,q}(A)+s)\right) \\
&> g\left(2(I_{g,q}(C)+s)\right) - \frac{1}{2} g\left(2(I_{g,q}(C)+s)\right) \\
&\geq \frac{1}{2}\left(2(I_{g,q}(C)+s)\right) = I_{g,q}(C)+s.
\end{aligned}
$$

Therefore since $\mathbf{x} \in F$, $x_i \leq 2(I_{g,q}(A)+s)$ for every i, which implicates that (**??**) holds. Thus we have

$$
S_{g,q}(I_{g,q}(A)+s) \subset H,
$$

from which we deduce that

$$
\begin{aligned}
Volume\left(A \cap S_{g,q}(I_{g,q}(A)+s)\right) &\leq Volume\left(S_{g,q}(I_{g,q}(A)+s)\right) \\
&\leq Volume(H) = 2^n (I_{g,q}(A)+s)^n.
\end{aligned}
$$

With this inequality, the upper bound of integration (22) can be given when $a_n \to \infty$ through

$$
\begin{aligned}
&\log \int_C \exp\left(-\sum_{1\leq i\leq n} (g(x_i)+q(x_i))\right) dx_1, ..., dx_n \\
&= -I_{g,q}(A) + \log \int_0^\infty Volume(A \cap S_{g,q}(I_{g,q}(A)+s))e^{-s} ds \\
&\leq -I_{g,q}(A) + \log \int_0^\infty \left(I_{g,q}(A)+s\right)^n e^{-s} ds + n\log 2.
\end{aligned}
$$

According to (21), it holds

$$
\int_0^\infty \left(I_{g,q}(A)+s\right)^n e^{-s} ds \leq (n+1) I_{g,q}(A)^n,
$$

Hence we have

$$
\begin{aligned}
&\log \int_A \exp\left(-\sum_{1\leq i\leq n} (g(x_i)+q(x_i))\right) dx_1, ..., dx_n \\
&\leq -I_{g,q}(A) + \log\left((n+1)I_{g,q}(A)^n\right) + n\log 2 \\
&= -I_{g,q}(A) + n\log I_{g,q}(A) + \log(n+1) + n\log 2.
\end{aligned}
$$

Replacing A by $I^c \cap C$ yields (38).

Step 4: In this step, we derive crude bounds for $I_{g_2}(C), I_{g_1}(I^c \cap C)$ and $I_{g_2}(I^c \cap C)$.

From (26) and (27), there exists some $a_n \in [X, \infty)$ (say, $a_n > y_2$) such that

$$M(x) \leq \max(N \log g(a_n), N \log g(x)) \tag{39}$$

holds on $(0, \infty)$. Hence for a_n large enough

$$g_2(x) = g(x) + M(x) \leq g(x) + \max(N \log g(a_n), N \log g(x)),$$

which in turn yields

$$I_{g_2}(C) \leq \inf_{\mathbf{x} \in C} \left(\sum_{i=1}^{n} g(x_i) + \sum_{i=1}^{n} \max(N \log g(a_n), N \log g(x_i)) \right). \tag{40}$$

It holds

$$\inf_{\mathbf{x} \in C} \left(\sum_{i=1}^{n} \max(N \log g(a_n), N \log g(x_i)) \right) = nN \log g(a_n) \tag{41}$$

which implies that

$$\begin{aligned}
&\inf_{\mathbf{x} \in C} \left(\sum_{i=1}^{n} g(x_i) + \sum_{i=1}^{n} \max(N \log g(a_n), N \log g(x_i)) \right) \\
&= \inf_{\mathbf{x} \in C} \left(\sum_{i=1}^{n} g(x_i) \right) + \inf_{\mathbf{x} \in C} \left(\sum_{i=1}^{n} \max(N \log g(a_n), N \log g(x_i)) \right) \\
&= \inf_{\mathbf{x} \in C} \left(\sum_{i=1}^{n} g(x_i) \right) + nN \log g(a_n) \\
&= I_g(C) + nN \log g(a_n) = ng(a_n) + nN \log g(a_n).
\end{aligned}$$

Thus we obtain the inequality

$$I_{g_2}(C) \leq ng(a_n) + nN \log g(a_n). \tag{42}$$

We now provide a lower bound of $I_{g_1}(I^c \cap C)$. Consider the inequality of (31) in **Step 1**, where we have showed that h is convex for x large enough; hence, using (31) when a_n is sufficiently large, it holds

$$I_{g_1}(I^c \cap C) \geq I_h(I^c \cap C) = \min \left(F_{h_1}(a_n, \epsilon_n), F_{h_2}(a_n, \epsilon_n) \right),$$

where the second inequality holds from Lemma 1. By the definition of the function h in (29), for large x it holds $h(x) = r(x)$ which yields the following lower bound of $I_{g_1}(I^c \cap C)$

$$I_{g_1}(I^c \cap C) \geq I_h(I^c \cap C) = I_r(I^c \cap C) = \min\left(F_{r_1}(a_n, \epsilon_n), F_{r_2}(a_n, \epsilon_n)\right).$$

By Lemma 1, it holds

$$\begin{aligned}
F_{r_1}(a_n, \epsilon_n) = g(a_n + \epsilon_n) + (n-1)g\left(a_n - \frac{1}{n-1}\epsilon_n\right) \\
- N\log g(a_n+\epsilon_n) - (n-1)N\log g\left(a_n - \frac{1}{n-1}\epsilon_n\right) \\
\geq g(a_n+\epsilon_n) + (n-1)g\left(a_n - \frac{1}{n-1}\epsilon_n\right) - nN\log g\left(a_n+\epsilon_n\right),
\end{aligned}$$

by the same way, we have also

$$F_{r_2}(a_n, \epsilon_n) \geq g(a_n - \epsilon_n) + (n-1)g\left(a_n + \frac{1}{n-1}\epsilon_n\right) - nN\log g\left(a_n+\epsilon_n\right),$$

hence

$$I_{g_1}(I^c \cap C) \geq \min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right) - nN\log g\left(a_n+\epsilon_n\right)$$

holds.

The method of the estimation of the upper bound of $I_{g_1}(I^c \cap C)$ is similar to that used for $I_{g_1}(C)$ above. In (40), replace C by $I^c \cap C$; we obtain

$$\begin{aligned}
I_{g_2}(I^c \cap C) &\leq \inf_{\mathbf{x} \in I^c \cap C}\left(\sum_{i=1}^{n} g(x_i) + \sum_{i=1}^{n}\max(N\log g(a_n), N\log g(x_i))\right) \\
&\leq \inf_{\mathbf{x} \in I^c \cap C}\left(\sum_{i=1}^{n} g(x_i) + \sum_{i=1}^{n}\max\left(N\log g\left(a_n + \frac{\epsilon_n}{n-1}\right), N\log g(x_i)\right)\right).
\end{aligned}$$

Similarly to (41), it holds

$$\inf_{\mathbf{x} \in I^c \cap C}\left(\sum_{i=1}^{n}\max\left(N\log g\left(a_n + \frac{\epsilon_n}{n-1}\right), N\log g(x_i)\right)\right) = nN\log g\left(a_n + \frac{\epsilon_n}{n-1}\right),$$

where equality is attained setting $x_1 = ... = x_{n-1} = a_n + \epsilon_n/(n-1), x_n =$

$a_n - \epsilon_n$. Hence we have, when $n \to \infty$

$$
\begin{aligned}
I_{g_2}(I^c \cap C) &\le \inf_{\mathbf{x} \in I^c \cap C} \left(\sum_{i=1}^n g(x_i) + \sum_{i=1}^n \max(N \log g\left(a_n + \frac{\epsilon_n}{n-1}\right), N \log g(x_i)) \right) \\
&= \inf_{\mathbf{x} \in I^c \cap C} \sum_{i=1}^n g(x_i) + nN \log g\left(a_n + \frac{\epsilon_n}{n-1}\right) \\
&= I_g(I^c \cap C) + nN \log g\left(a_n + \frac{\epsilon_n}{n-1}\right) \\
&\le g(a_n - \epsilon_n) + (n-1) g\left(a_n + \frac{1}{n-1}\epsilon_n\right) + nN \log g\left(a_n + \frac{\epsilon_n}{n-1}\right) \\
&\le ng\left(a_n + \frac{\epsilon_n}{n-1}\right) + nN \log g\left(a_n + \frac{\epsilon_n}{n-1}\right) \\
&\le n(N+1) g\left(a_n + \frac{\epsilon_n}{n-1}\right).
\end{aligned}
$$

Therefore we obtain

$$
\log I_{g_2}(I^c \cap C) \le \log n + \log(N+1) + \log g\left(a_n + \frac{\epsilon_n}{n-1}\right). \tag{43}
$$

Step 5: In this step, we complete the proof by showing that

$$
\lim_{a_n \to \infty} \frac{P(I^c \cap C)}{P(C)} = 0.
$$

Using the upper bound of $P(I^c \cap C)$, together with the lower bound of $P(C)$ above, we have under condition (11) when a_n is large enough

$$
\begin{aligned}
\frac{P(I^c \cap C)}{P(C)} &\le \exp\begin{pmatrix} -\left(I_{g,q}(I^c \cap C) - I_{g,q}(C)\right) + n \log I_{g,q}(I^c \cap C) \\ +\tau_n + n \log g(a_n) + \log(n+1) + n \log 2 \end{pmatrix} \\
&\le \exp\left(-\left(I_{g,q}(I^c \cap C) - I_{g,q}(C)\right) + n \log I_{g,q}(I^c \cap C) + \tau_n + 2n \log g(a_n)\right) \\
&\le \exp\left(-\left(I_{g_1}(I^c \cap C) - I_{g_2}(C)\right) + n \log I_{g_2}(I^c \cap C) + \tau_n + 2n \log g(a_n)\right).
\end{aligned}
$$

The last inequality holds from (24) and (25). Replace $I_{g_1}(I^c \cap C), I_{g_2}(C)$ by the upper bound of (42) and the lower bound of (**??**), respectively, we obtain

$$
\begin{aligned}
I_{g_1}(I^c \cap C) - I_{g_2}(C) &\ge \min\left(F_{g_1}(a_n, \epsilon_n), F_{g_2}(a_n, \epsilon_n)\right) - nN \log g\left(a_n + \epsilon_n\right) \\
&\qquad - \left(ng(a_n) + nN \log g(a_n)\right) \\
&= H(a_n, \epsilon_n) - nN \log g\left(a_n + \epsilon_n\right) - nN \log g(a_n) \\
&\ge H(a_n, \epsilon_n) - 2nN \log\left(a_n + \epsilon_n\right).
\end{aligned} \tag{44}
$$

Under condition (11), there exists some Q such that $n \log n \leq Qn \log g(a_n)$, which, together with (43) and (44), gives

$$\begin{aligned}
\frac{P(I^c \cap C)}{P(C)} &\leq \exp \begin{pmatrix} -\left(H(a_n, \epsilon_n) - 2nN \log\left(a_n + \epsilon_n\right)\right) + n \log n + n \log(N+1) \\ +n \log g\left(a_n + \frac{\epsilon_n}{n-1}\right) + \tau_n + 2n \log g(a_n) \end{pmatrix} \\
&= \exp \begin{pmatrix} -H(a_n, \epsilon_n) + n(2N+1) \log g\left(a_n + \epsilon_n\right) \\ +\tau_n + 2n \log g(a_n) + n \log n + n \log(N+1) \end{pmatrix} \\
&\leq \exp\left(-H(a_n, \epsilon_n) + n(2N+1) \log g\left(a_n + \epsilon_n\right) + \tau_n + 2n \log g(a_n) + 2n \log n\right) \\
&\leq \exp\left(-H(a_n, \epsilon_n) + n(2N+1) \log g\left(a_n + \epsilon_n\right) + \tau_n + (2Q+2)n \log g(a_n)\right) \\
&\leq \exp\left(-H(a_n, \epsilon_n) + n(2N+2Q+3) \log g\left(a_n + \epsilon_n\right) + \tau_n\right).
\end{aligned} \tag{45}$$

The second term in the bracket in the last line above and τ_n are both of small order with respect to $H(a_n, \epsilon_n)$. Indeed under condition (12), when $a_n \to \infty$, it holds

$$\lim_{n \to \infty} \frac{n(2N+2Q+3) \log g\left(a_n + \frac{\epsilon_n}{n-1}\right)}{H(a_n, \epsilon_n)} = 0. \tag{46}$$

For τ_n which is defined in (36)under conditions (12), (13), $nN \log g(a_n)$ and $nG(a_n)$ are both of smaller order than $H(a_n, \epsilon_n)$. As regards to the third term of τ_n, it holds

$$\begin{aligned}
nN \log g\left(a_n + \frac{1}{g(a_n)}\right) &= nN \log\left(g\left(a_n + \frac{1}{g(a_n)}\right) - g(a_n) + g(a_n)\right) \\
&\leq nN \log\left(2 \max\left(G(a_n), g(a_n)\right)\right) \\
&= nN \log 2 + \max\left(nN \log G(a_n), nN \log g(a_n)\right).
\end{aligned}$$

Under conditions (12) and (13), both $nN \log G(a_n)$ and $nN \log g(a_n)$ are small with respect to $H(a_n, \epsilon_n)$; therefore $nN \log g\left(a_n + 1/g(a_n)\right)$ is small with respect to $H(a_n, \epsilon_n)$ when $a_n \to \infty$. Hence it holds when $a_n \to \infty$

$$\lim_{n \to \infty} \frac{\tau_n}{H(a_n, \epsilon_n)} = 0.$$

Finally, (45), together with (46) and (**??**), implies that (**??**) holds.

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